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THE STRUCTURE OF MULTIVARIATE SUPERSPLINE SPACES OF HIGH DEGREE

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ABSTRACT. We consider splines (of global smoothness r, polynomial degree d, in a general number k of independent variables, defined on a k-dimensional triangulation $\mathscr{T}$ of a suitable domain Ω) which are $r2^{k-m-1}$-times differentiable across every m-face ($m = 0, \cdots, k-1$) of a simplex in $\mathscr{T}$. For the case $d > r2^k$ we identify a structure that allows the construction of a minimally supported basis.

1. INTRODUCTION

A *κ-simplex* K $(0 \le \kappa \le k)$ is the convex hull of $\kappa + 1$ points in $\mathbb{R}^k$ called the *vertices* of K. K is *nondegenerate* if its κ-dimensional volume is nonzero, and *degenerate* otherwise. The *dimension* of a nondegenerate κ-simplex is κ. The convex hull of a subset of $\mu + 1$ vertices of K is a μ-face of K.

Let $\mathscr{V} \subset \mathbb{R}^k$ be a given set of N distinct points.

A *triangulation* $\mathscr{T}$ of the set $\mathscr{V}$ is a set of nondegenerate k-simplices satisfying the following requirements:

1. All vertices of each simplex in $\mathscr{T}$ are elements of $\mathscr{V}$.
2. The interiors of the simplices in $\mathscr{T}$ are pairwise disjoint.
3. The set

$$\Omega := \bigcup_{K \in \mathscr{T}} K \subset \mathbb{R}^k \tag{1}$$

   is homeomorphic to $[0, 1]^k$.
4. Each $(k-1)$-face of a simplex in $\mathscr{T}$ is either on the boundary of Ω, or else is a common face of exactly two simplices in $\mathscr{T}$.
5. No simplex in $\mathscr{T}$ contains any points of $\mathscr{V}$ other than its vertices.

Note that a μ-face of a simplex in $\mathscr{T}$ is itself a μ-dimensional simplex. On the triangulation $\mathscr{T}$ we define a spline space $S_d^r(\mathscr{T})$ as usual by

$$S_d^r(\mathscr{T}) = \{s \in C^r(\Omega) : s|_\tau \in \mathscr{P}_d^k \ \forall \tau \in \mathscr{T}\}, \tag{2}$$

where $\mathscr{P}_d^k$ is the $\binom{k+d}{d}$-dimensional linear space of all k-variate polynomials of total degree less than or equal to d.

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In this paper, we consider subspaces of $S_d^r(\mathscr{T})$ obtained by increasing the smoothness requirements across faces of the underlying simplices. More precisely, denoting by $\mathscr{S}_\mu$ the set of all μ-faces of the simplices in $\mathscr{T}$ ($\mu = 0, \dots, k-1$) and letting $\mathscr{S} = \bigcup_{\mu=0}^{k-1} \mathscr{S}_\mu$, we define the (superspline) space $\mathbb{S}_d^r(\mathscr{T})$ as a subspace of $S_d^r(\mathscr{T})$ as follows:

$$\mathbb{S}_d^r(\mathscr{T}) = \{s \in S_d^r(\mathscr{T}) : s \text{ is } \rho\text{-times differentiable across } \sigma \ \forall \sigma \in \mathscr{S}\}, \quad \text{where } \rho = r2^{k-\dim\sigma-1}. \tag{3}$$

The concept of supersplines was introduced in Chui and Lai [8], [9]. The area in between finite elements and full spline spaces was further explored by Schumaker [16] and Ibrahim and Schumaker [11].

2. The generalized Bézier-Bernstein form

Crucial to analyzing the dimension of spline spaces is the Bézier-Bernstein form of a multivariate polynomial. In the case $k \le 2$ this form is used widely and is well known. A review of the Bézier-Bernstein form for a general number of variables is in de Boor [6]. In this paper, we use a notation that is particularly suitable for our purposes. However, generalized barycentric coordinates and global control nets have also been proposed in Alfeld [2] and de Boor [6].

We use $\mathscr{V}$ as an index set and denote by $\mathbb{N}$ the set of nonnegative integers. For vectors $\mathbf{I} = [i_v]_{v\in\mathscr{V}} \in \mathbb{N}^N$ and $\mathbf{a} = [a_v]_{v\in\mathscr{V}} \in \mathbb{R}^N$ we define

$$|\mathbf{I}| = \sum_{v\in\mathscr{V}} i_v, \tag{4}$$

$$\mathbf{a}^{\mathbf{I}} = |\mathbf{I}|! \prod_{v\in\mathscr{V}} a_v^{i_v} \Big/ \prod_{v\in\mathscr{V}} i_v!, \tag{5}$$

where

$$0^0 := 1. \tag{6}$$

We also use the notation

$$\sigma(\mathbf{I}) = \operatorname{conv}\{v : i_v > 0\}, \qquad \sigma(\mathbf{a}) = \operatorname{conv}\{v : a_v \neq 0\} \tag{7}$$

where $\operatorname{conv} X$ denotes the convex hull of a point set X.

We now define *generalized barycentric coordinates* as cardinal piecewise linear functions $b_v \in S_1^0(\Omega)$ by the requirement

$$b_v(w) = \delta_{vw} = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{else} \end{cases} \quad \forall v, w \in \mathscr{V}. \tag{8}$$

Clearly, in each k-simplex $K \in \mathscr{T}$ the functions b_v, where v is a vertex of K, reduce to the ordinary barycentric coordinates. Globally, i.e., for all $x \in \Omega$, they satisfy

$$\sum_{v\in\mathscr{V}} b_v = 1, \quad b_v \ge 0 \ \ \forall v \in \mathscr{V}, \quad \text{and} \quad x = \sum_{v\in\mathscr{V}} b_v(x) v. \tag{9}$$

For a given polynomial degree d, we use the *domain index set*

$$\mathbf{I}_d = \{\mathbf{I} \in \mathbb{N}^N : |\mathbf{I}| = d \text{ and } \sigma(\mathbf{I}) \in \mathscr{S} \cup \mathscr{T}\}. \tag{10}$$

Letting

$$\mathbf{b} = \mathbf{b}(x) = [b_v(x)]_{v \in \mathscr{V}}, \tag{11}$$

it is clear that every function $s \in S_d^0(\mathscr{T})$ can be written as

$$s = \sum_{\mathbf{I} \in \mathbf{I}_d} c_{\mathbf{I}} \mathbf{b}^{\mathbf{I}}. \tag{12}$$

The coefficients $c_{\mathbf{I}}$ are the *Bézier ordinates* of s.

3. The de Casteljau algorithm

Let $s \in S_d^0(\mathscr{T})$, $K \in \mathscr{T}$, and $s|_K = p \in \mathscr{P}_d^k$. Without loss of generality we may relabel the vertices and assume that K is the k-simplex

$$K = \{\mathbf{a} = (a_1, \cdots, a_{k+1}) : a_1 + \cdots + a_{k+1} = d,\ a_j \geq 0,\ a_j \in \mathbb{R}\}. \tag{13}$$

Then, with V_j denoting the vertices of K, we have

$$p = \sum_{\mathbf{I} \in K \cap \mathbf{I}_d} c_{\mathbf{I}} \mathbf{b}^{\mathbf{I}} \quad \text{where} \quad x = \sum_{j=1}^{k+1} b_j V_j, \quad \sum_{j=1}^{k+1} b_j = 1. \tag{14}$$

The Bernstein polynomials $\mathbf{b}^{\mathbf{I}}(x)$ satisfy a recurrence relation, see Farin [10]

$$\mathbf{b}^{\mathbf{I}}(x) = \sum_{j=1}^{k+1} b_j \mathbf{b}^{\mathbf{I}-\varepsilon^j}(x), \quad |\mathbf{I}| = d, \quad \text{and} \quad \varepsilon^j = (0, \cdots, 1, \cdots, 0). \tag{15}$$

This relation allows one to expand p in terms of Bernstein polynomials of lower degree with (polynomial) coefficients $p_{\mathbf{I}}^r(\mathbf{b})$.

Theorem 1. *We have*

$$p = \sum_{|\mathbf{I}|=d-r} p_{\mathbf{I}}^r(\mathbf{b}) \mathbf{b}^{\mathbf{I}}, \qquad 0 \leq r \leq d, \tag{16}$$

where

$$p_{\mathbf{I}}^0(\mathbf{b}) = c_{\mathbf{I}}, \tag{17}$$

$$p_{\mathbf{I}}^r(\mathbf{b}) = \sum_{j=1}^{k+1} b_j p_{\mathbf{I}+\varepsilon^j}^{r-1}(\mathbf{b}), \qquad |\mathbf{I}| = d - r, \qquad 0 \leq r \leq d. \tag{18}$$

The intermediate coefficients $p_{\mathbf{I}}^r(\mathbf{b})$ can also be written explicitly as

$$p_{\mathbf{I}}^r(\mathbf{b}) = \sum_{|\mathbf{M}|=r} c_{\mathbf{I}+M} \mathbf{b}^{\mathbf{M}}, \qquad |\mathbf{I}| = d - r. \tag{19}$$

The formulas in Theorem 1 may be used to evaluate p at a given point, and are referred to as the *de Casteljau Algorithm*.

4. Smoothness across an interface

Given a vector $e \in \mathbb{R}^k$, the directional derivative of p in the direction of $\alpha = (\frac{\partial b_1}{\partial e}, \dots, \frac{\partial b_{k+1}}{\partial e})$, denoted D_α, is given by (Alfeld [2])

$$D_\alpha p = d \sum_{|\mathbf{I}|=d-1} p_{\mathbf{I}}^1(\alpha)\mathbf{b}^{\mathbf{I}}, \tag{20}$$

where

$$p_{\mathbf{I}}^1(\alpha) = \sum_{|\mathbf{M}|=1} c_{\mathbf{I}+M}\boldsymbol{\alpha}^{\mathbf{M}}, \qquad |\mathbf{I}| = d-1. \tag{21}$$

In general, the rth directional derivative of p in the direction of α is given by (Farin [10])

$$D_\alpha^r p = \frac{d!}{(d-r)!} \sum_{|\mathbf{I}|=d-r} p_{\mathbf{I}}^r(\alpha)\mathbf{b}^{\mathbf{I}}, \tag{22}$$

where

$$p_{\mathbf{I}}^r(\alpha) = \sum_{|\mathbf{M}|=r} c_{\mathbf{I}+M}\boldsymbol{\alpha}^{\mathbf{M}}, \quad |\mathbf{I}| = d-r, \quad \text{and} \quad p_{\mathbf{I}}^0(\alpha) = c_{\mathbf{I}}. \tag{23}$$

The following theorem was proved by Farin [10] in the bivariate case; here we state without proof the result in any number of variables. Let τ be an m-face of K; without loss of generality, we will assume that

$$\tau = \{\mathbf{a} \in K\colon\ a_{m+2} = \cdots = a_{k+1} = 0\}. \tag{24}$$

Theorem 2. *Let $p, q \in \mathscr{P}_d^k$ be such that $p|_\tau = q|_\tau$. Then $D_\alpha^s p = D_\alpha^s q$ on τ for all directions α, $0 \le s \le r$, if and only if $p_{\mathbf{J}}^r = q_{\mathbf{J}}^r$ for all $\mathbf{J} = (j_1, \dots, j_{m+1}, 0, \dots, 0)$, $|\mathbf{J}| = d-r$.*

5. Subsimplices and subpolynomials

Let $K \in \mathscr{T}$ be a k-simplex and τ be a face of K with $\dim \tau = n$,

$$K = \left\{\mathbf{a} \in \mathbb{R}^N\colon \sum_{v \in K} a_v = d,\ a_v \ge 0\right\}, \tag{25}$$

$$\tau = \left\{\mathbf{a} \in K\colon \sum_{v \in \tau} a_v = d\right\}. \tag{26}$$

For $\mathbf{J} \in K \cap \mathbf{I}_d$, so that $\sum_{v \in \tau} j_v = d - \rho$, define

$$\tau_{\mathbf{J}} = \{\mathbf{a} = [a_v]_{v \in K}\colon\ j_v \ge a_v,\ v \in \tau\}. \tag{27}$$

Clearly, $\tau_{\mathbf{J}}$ is a subsimplex of K similar to K.

Let $p \in \mathscr{P}_d^k$, and $(\mathbf{I}, c_\mathbf{I})_{\mathbf{I}\in K}$ be the control points of p with respect to K. The control points $(\mathbf{I}, c_\mathbf{I})_{\mathbf{I}\in\tau_\mathbf{J}}$ define a polynomial $p_\mathbf{J}$ of degree ρ. We call $\tau_\mathbf{J}$ the subsimplex of K associated with $\mathbf{J}$, and $p_\mathbf{J}$ the subpolynomial of p associated with $\mathbf{J}$.

De Boor [6] introduces the concepts of subsimplices and subpolynomials and proves most of the results of this section. Since our definitions are slightly different, we restate two key facts involving subpolynomials. Theorem 2 can be restated as follows:

Theorem 3. *Let* $p, q \in \mathscr{P}_d^k$, *and* τ *be a face of* K *with* $\dim\tau = m$ *such that* $p|_\tau = q|_\tau$. *Then*

$$D_\alpha^s p = D_\alpha^s q \text{ on } \tau, \text{ for all directions } \alpha \text{ and } 0 \le s \le \rho, \tag{28}$$

if and only if

$$p_\mathbf{I} = q_\mathbf{I} \quad \forall \mathbf{I} \in K \text{ with } \operatorname{dist}(\mathbf{I}, \tau) = \rho. \tag{29}$$

The above theorem immediately yields the following:

Theorem 4. *Let* $K, K' \in \mathscr{T}$, $\tau \subset K \cap K'$, $p, q \in \mathscr{P}_d^k$, *and let* $s \in S_d^0(\Omega)$ *with* $s|_K = p$, $s|_{K'} = q$. *Then* $s \in C^\rho(\tau)$ *if and only if* $p_\mathbf{I} = q_\mathbf{I}$ $\forall \mathbf{I} \in K'$ *with* $\operatorname{dist}(\mathbf{I}, \tau) = \rho$.

6. Determining sets

We now generalize the concept of a *determining set* known from the bivariate case (see Alfeld and Schumaker [4]).

Definition 5. A set $D \subset \mathbf{I}_d$ is a determining set of $\mathbb{S}_d^r(\mathscr{T})$ if, for all $s \in \mathbb{S}_d^r(\mathscr{T})$,

$$c_\mathbf{I} = 0 \qquad \forall \mathbf{I} \in D \Longrightarrow s \equiv 0. \tag{30}$$

D is a minimal determining set if there is no determining set which has fewer elements than D.

It is clear from elementary linear algebra that the number of elements in a determining set of $\mathbb{S}_d^r(\mathscr{T})$ provides an upper bound on the dimension of $\mathbb{S}_d^r(\mathscr{T})$, and that the number of elements in a minimal determining set is unique and equals the dimension of $\mathbb{S}_d^r(\mathscr{T})$.

7. A minimal determining set of $\mathbb{S}_d^r(\mathscr{T})$

For each simplex $\sigma \in \mathscr{S} \cup \mathscr{T}$ let $\rho = r2^{k-\dim\sigma-1}$ as before; we define two sets of domain indices recursively by

$$\overline{\mathscr{D}}(\sigma) = \left\{\mathbf{I} \in \mathbf{I}_d : \sum_{v\in\sigma} i_v \ge d - \rho\right\} \tag{31}$$

and

$$\mathscr{D}(\sigma) = \overline{\mathscr{D}}(\sigma) \setminus \bigcup_{\tau\prec\sigma} \mathscr{D}(\tau), \tag{32}$$

where τ is a proper face of σ.

Definition 6. Let $\sigma \in \mathcal{S} \cup \mathcal{T}$. A set $\mathcal{A}(\sigma) \subset D(\sigma)$ is a determining set of $\overline{\mathcal{D}}(\sigma)$ if, for all $s \in \mathbb{S}_d^r(\mathcal{T})$,

$$c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)) \Longrightarrow c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \overline{\mathcal{D}}(\sigma). \tag{33}$$

The set $\mathcal{A}$ is a minimal determining set of $\overline{\mathcal{D}}(\sigma)$ if there is no determining set of $\overline{\mathcal{D}}(\sigma)$ with fewer elements.

Lemma 7. *Let $d > r2^k$. Then, for all $\mathbf{I} \in I_d$ there exists a unique $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that $\mathbf{I} \in \mathcal{D}(\sigma)$.*

Proof. To prove the lemma, we have to show first that for all $\sigma, \tau \in \mathcal{S} \cup \mathcal{T}$:

$$\sigma \neq \tau \Longrightarrow \mathcal{D}(\sigma) \cap \mathcal{D}(\tau) = \emptyset. \tag{34}$$

To establish this, suppose there is a domain index $\mathbf{I} \in \mathcal{D}(\sigma) \cap \mathcal{D}(\tau)$, for two simplices $\sigma, \tau \in \mathcal{S} \cup \mathcal{T}$, such that $\dim \sigma \geq \dim \tau$, and τ is not a face of σ. Thus,

$$\sum_{v \in \sigma} i_v \geq d - r2^{k - \dim \sigma - 1} \quad \text{and} \quad \sum_{v \in \tau} i_v \geq d - r2^{k - \dim \tau - 1}, \tag{35}$$

which implies

$$\sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v \geq 2d - r2^{k - \dim \sigma - 1} - r2^{k - \dim \tau - 1}. \tag{36}$$

Moreover,

$$\sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v = \sum_{v \in \sigma \cap \tau} i_v + \sum_{v \in \sigma \cup \tau} i_v. \tag{37}$$

Now, rearranging (37), substituting (36) into (37), and using $\sum_{v \in \sigma \cup \tau} i_v \leq d$, we obtain

$$\sum_{v \in \sigma \cap \tau} i_v \geq d - r2^{k - \dim \tau}. \tag{38}$$

So there exists a face $\tilde{\tau}$ of τ such that $\mathbf{I} \in \mathcal{D}(\tilde{\tau})$, which is a contradiction.

Finally, we need to prove that $\mathbf{I}_d = \bigcup_{\sigma \in \mathcal{S} \cup \mathcal{T}} \mathcal{D}(\sigma)$.

To see this, we only need to show that for each domain index $\mathbf{I} \in \mathbf{I}_d$ there exists a simplex $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that $\mathbf{I} \in \mathcal{D}(\sigma)$. However, this is trivial, since for each domain index $\mathbf{I}$ there exists at least one k-simplex K such that $\sum_{v \in K} i_v = d$. Thus, $\mathbf{I} \in \overline{\mathcal{D}}(K)$, and there must be a simplex $\sigma \prec K$ such that $\mathbf{I} \in \mathcal{D}(\sigma)$. □

Lemma 8. *Let $d > r2^k$, $\rho = r2^{k - \dim \sigma - 1}$, and $\mathcal{A}(\sigma) := \mathcal{D}(\sigma) \cap K$, where $K \in \mathcal{T}$ and $\sigma \prec K$. Then $\mathcal{A}(\sigma)$ is a minimal determining set of $\overline{\mathcal{D}}(\sigma)$.*

Proof. First we establish that $\mathcal{A}(\sigma)$ is determining.

Let $s \in \mathbb{S}_d^r(\mathcal{T})$, and let $s|_K = p$, with $p \in \mathcal{P}_d^k$. If

$$c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)),$$

then

$$p_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)),$$

where $p_{\mathbf{I}}$ is the subpolynomial of p of degree $\leq \rho$ associated with $\mathbf{I}$ (notice that $\operatorname{dist}(\mathbf{I}, \sigma) \leq \rho \quad \forall \mathbf{I} \in \mathscr{A}(\sigma)$). Let K' be any other k-simplex of $\mathscr{T}$ such that $\sigma \prec K'$ and $K \neq K'$, and let $q = s|_{K'}$, $q \in \mathscr{P}_d^k$. Now, $s \in C^{\rho}(\sigma)$; then $\forall \mathbf{I} \in (\mathscr{D}(\sigma) \cap K') \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)) p_{\mathbf{I}} = q_{\mathbf{I}}$, where $q_{\mathbf{I}}$ is the subpolynomial of q of degree $\leq \rho$ associated with $\mathbf{I}$. But,

$$p_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in (\mathscr{D}(\sigma) \cap K) \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)) \tag{39}$$

$$\Longrightarrow \quad q_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in (\mathscr{D}(\sigma) \cap K') \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)) \tag{40}$$

$$\Longrightarrow \quad c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in (\mathscr{D}(\sigma) \cap K') \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)) \tag{41}$$

$$\Longrightarrow \quad c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \left(\bigcup_{\substack{K \in \mathscr{T} \\ \sigma \prec K}} \mathscr{D}(\sigma) \cap K \right) \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)) \tag{42}$$

$$\Longrightarrow \quad c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathscr{D}(\sigma) \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)). \tag{43}$$

Therefore,

$$c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \overline{\mathscr{D}}(\sigma). \tag{44}$$

To see that $\mathscr{A}(\sigma)$ is indeed minimal, take $\mathbf{I} \in \mathscr{A}(\sigma)$ and consider the set $\widetilde{\mathscr{A}}(\sigma) = \mathscr{A}(\sigma) \setminus \{\mathbf{I}\}$, define a polynomial on K whose Bézier coefficients are equal to zero except at the domain point $\mathbf{I}$, where $c_{\mathbf{I}} = 1$, and extend this polynomial globally on the rest of the triangulation.

The coefficients $c_{\mathbf{J}}$ are equal to zero on $\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma)$, since the smoothness conditions there only involve domain indices in $\mathscr{D}(\tau)$ for $\tau \prec \sigma$.

Hence, $c_{\mathbf{J}} = 0 \quad \forall J \in \widetilde{\mathscr{A}}(\sigma) \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma))$, but this does not imply that $c_{\mathbf{J}} = 0 \quad \forall J \in \overline{\mathscr{D}}(\sigma)$ since in particular $I \in \overline{\mathscr{D}}(\sigma)$ and $c_{\mathbf{I}} = 1$.

Therefore, $\widetilde{\mathscr{A}}(\sigma)$ cannot be determining and $\mathscr{A}(\sigma)$ is a minimal determining set of $\overline{\mathscr{D}}(\sigma)$. □

The following theorem is the central result of this paper.

Theorem 9. *Let $r \geq 0$, $d > r2^k$ and let $\mathscr{A}(\sigma) = \mathscr{D}(\sigma) \cap K$, where $K \in \mathscr{T}$ is a k-simplex so that σ is a face of K. Then*

$$\mathscr{A} := \bigcup_{\sigma \in \mathscr{S} \cup T} \mathscr{A}(\sigma) \tag{45}$$

is a minimal determining set of $\mathbb{S}_d^r(\mathscr{T})$.

Proof. Let $s \in \mathbb{S}_d^r(\mathscr{T})$ and assume that $c_{\mathbf{I}} = 0 \quad \forall I \in \mathscr{A}$. Using induction on $\dim \sigma$, we first establish that $\mathscr{A}$ is a determining set.

(1) If $\dim \sigma = 0$ then $c_{\mathbf{I}} = 0 \quad \forall I \in \mathscr{A}(\sigma)$ implies $c_{\mathbf{I}} = 0 \quad \forall I \in \overline{\mathscr{D}}(\sigma)$, since $\mathscr{D}(\sigma) = \overline{\mathscr{D}}(\sigma)$.

(2) We assume now that $c_{\mathbf{I}} = 0 \quad \forall I \in \overline{\mathscr{D}}(\sigma)$ and $\forall \sigma$ with $\dim \sigma < n$.

(3) Let $\dim\sigma = n$, $c_{\mathbf{I}} = 0$ $\forall I \in \mathscr{A}(\sigma)$. Let $\mathscr{B}(\sigma) = \mathscr{A}(\sigma) \cup (\overline{\mathscr{D}}(\sigma) \setminus \mathscr{D}(\sigma))$; notice that $B(\sigma) \subset \mathscr{A}(\sigma) \cup (\bigcup_{\tau \prec \sigma} \overline{\mathscr{D}}(\tau))$.

By the induction hypothesis, $c_{\mathbf{I}} = 0$ $\forall I \in \bigcup_{\tau \prec \sigma} \overline{\mathscr{D}}(\tau)$ implies that $c_{\mathbf{I}} = 0$ $\forall I \in \mathscr{B}(\sigma)$. But $\mathscr{A}(\sigma)$ is a determining set of $\overline{\mathscr{D}}(\sigma)$; thus, $c_{\mathbf{I}} = 0$ $\forall I \in \overline{\mathscr{D}}(\sigma)$, hence

$$c_{\mathbf{I}} = 0 \quad \forall I \in \bigcup_{\sigma \in \mathscr{S} \cup T} \overline{\mathscr{D}}(\sigma), \tag{46}$$

and by Lemma 7 we get that

$$c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathbf{I}_d. \tag{47}$$

To show that $\mathscr{A}$ is indeed minimal, we give arbitrary Bézier ordinates $c_{\mathbf{I}}$ $\forall \mathbf{I} \in \mathscr{A}$, and we construct the rest of the Bézier ordinates in such a way that the piecewise polynomial they determine is a superspline.

So, we assume $c_{\mathbf{I}}$ $\forall \mathbf{I} \in \mathscr{A}$ are given, and let $\mathbf{J} \in \mathbf{I}_d$. Then by Lemma 7 there is a unique $\sigma \in \mathscr{S} \cup T$ such that $\mathbf{J} \in \mathscr{D}(\sigma)$.

We define $c_{\mathbf{J}}$ inductively over $\dim\sigma$.

(1) If σ is a vertex, then there exists $K \in \mathscr{T}$ so that $\sigma \prec K$ and $\mathscr{A}(\sigma) \subset K$.

Let $\sigma_{\mathbf{J}}$ be the subsimplex of K associated with $\mathbf{J}$, $\sigma_{\mathbf{J}} \subset \mathscr{A}(\sigma)$. Then the polynomial $p_{\mathbf{J}}$ is defined; so, for $\mathbf{J} \in \mathscr{D}(\sigma) \cap K'$ with $K' \in \mathscr{T}$, $K' \neq K$, let $\tau_{\mathbf{J}}$ be the subsimplex of K' associated with $\mathbf{J}$. Then there is a unique way to give Bézier ordinates to the domain points in $\tau_{\mathbf{J}}$ in such a way that they define a polynomial $q_{\mathbf{J}}$ which equals $p_{\mathbf{J}}$. Furthermore, in this manner, $c_{\mathbf{J}}$ can be defined for all $\mathbf{J} \in \mathscr{D}(\sigma) \setminus \mathscr{A}(\sigma)$.

(2) Suppose $c_{\mathbf{J}}$ has been defined for all $\mathbf{J} \in \bigcup_{\dim\sigma < n} \mathscr{D}(\sigma)$.

(3) Let $\mathbf{J} \in \mathscr{D}(\sigma)$ with σ an n-simplex and $K \in \mathscr{T}$ such that $\mathscr{A}(\sigma) \subset K$. And let again $\sigma_{\mathbf{J}}$ be the subsimplex of K associated with $\mathbf{J}$. Note that

$$\sigma_{\mathbf{J}} \subset \overline{\mathscr{D}}(\sigma) \cap K \subset \mathscr{A}(\sigma) \cup \left(\bigcup_{\tau \prec \sigma} \mathscr{D}(\tau) \cap K \right). \tag{48}$$

Then by the induction hypothesis and by the fact that the Bézier ordinates have been defined on $\mathscr{A}$, we have that all of the Bézier ordinates on $\sigma_{\mathbf{J}}$ are defined; therefore $p_{\mathbf{J}}$ is defined. So as before, for $\mathbf{J} \in \mathscr{D}(\sigma) \cap K'$, $K' \neq K$, we can define $q_{\mathbf{J}}$ in the same way we did when σ was a vertex. Thus, $c_{\mathbf{I}}$ has been defined for all $\mathbf{I} \in \mathbf{I}_d$.

Next, we need to show that the piecewise polynomial function defined by the $c_{\mathbf{I}}$'s is well defined.

Suppose $p_{\mathbf{L}} = q_{\mathbf{L}}$ for $\mathbf{L} \in K$ and $\sum_{v \in \xi} l_v = d - \rho$.

Let $\mathbf{J} \in \mathscr{D}(\sigma) \cap K'$ and suppose that $\mathbf{J} \in \xi_{\mathbf{L}}$ with $\mathbf{J} \neq \mathbf{L}$, where

$$\xi_{\mathbf{L}} = \{\mathbf{I} \in K' : i_v \geq l_v \ \ v \in \xi, \ \ \xi \in \mathscr{S} \cup T, \text{ and } \sigma \prec \xi\} \tag{49}$$

and

$$\tau_{\mathbf{J}} = \{\mathbf{I} \in K' : i_v \geq j_v \ \ v \in \sigma\}. \tag{50}$$

Then $\mathbf{J} \in \xi_{\mathbf{L}}$ implies $j_v \geq l_v$, which in turn implies $\tau_{\mathbf{J}} \subset \xi_{\mathbf{L}}$. Therefore, $q_{\mathbf{J}}$ is a subpolynomial of $q_{\mathbf{L}}$. Similarly, $p_{\mathbf{J}}$ is a subpolynomial of $p_{\mathbf{L}}$. Hence, $p_{\mathbf{J}} = q_{\mathbf{J}}$ since $q_{\mathbf{L}} \cup p_{\mathbf{L}} \in C^\infty(\xi)$. Therefore, the Bézier ordinates are well defined and by construction, the piecewise polynomial defined is a superspline. □

Corollary 10. *We have* $\dim \mathbb{S}_d^r(\mathscr{T}) = \sum_{\sigma \in \mathscr{S} \cup \mathscr{T}} |\mathscr{A}(\sigma)|$.

In view of the above corollary, to compute $\dim \mathbb{S}_d^r(\mathscr{T})$ we need only to know the cardinality of $\mathscr{A}(\sigma)$ for every $\sigma \in \mathscr{S}$. Clearly, $|\mathscr{A}(\sigma)|$ depends only on $m = \dim \sigma$. The following theorem is proved in Alfeld and Sirvent [5].

Theorem 11. *We have* $\dim \mathbb{S}_d^r(\mathscr{T}) = \sum_{m=0}^{k} \phi(m) f_m$, *where* f_m *is the number of m-simplices of* $\mathscr{S} \cup \mathscr{T}$, *and for* $m = 0, \ldots, k$, $\phi(m) = \phi_m^k(0)$, *where, with* $\rho_m = r2^{k-m-1}$, *the quantities* $\phi_q^m(p)$ *are defined recursively by*

$$\phi_q^m(p) = \sum_{j=0}^{\rho_q - p} \binom{j+m-q-1}{j} \left[\binom{d-p-j+q}{q} - \sum_{i=0}^{q-1} \binom{q+1}{i+1} \phi_i^q(p+j) \right] \tag{51}$$

if $0 < q < m$, *and*

$$\phi_0^m(p) = \sum_{j=0}^{\rho_0 - p} \binom{j+m-1}{j} = \binom{\rho_0 - p + m}{m}, \tag{52}$$

$$\phi_k^k(0) = \binom{d+k}{k} - \sum_{m=0}^{k-1} \binom{k+1}{m+1} \phi_m^k(0). \tag{53}$$

8. Minimally supported bases

Definition 12. The star of a simplex $\sigma \in \mathscr{S} \cup T$, denoted $\operatorname{star}(\sigma)$, is the set of all k-simplices $K \in \mathscr{T}$ such that σ is a face of K.

Definition 13. A basis $\{l_\mu : \mu = 1, 2, \ldots, \dim \mathbb{S}_d^r(\mathscr{T})\}$ is said to be minimally supported if for each basis function l_μ there exists a simplex $\sigma \in \mathscr{S} \cup T$ such that the support of l_μ is contained in $\operatorname{star}(\sigma)$.

The basis functions constructed in the proof of Theorem 9 are minimally supported: using the same construction, we can define cardinal supersplines

$$l_{\mathbf{I}} \in \mathbb{S}_d^r(\mathscr{T}) : \quad l_{\mathbf{I}}(\mathbf{J}) = \delta_{\mathbf{IJ}} \quad \forall \mathbf{I}, \mathbf{J} \in \mathscr{A} \tag{54}$$

so that, if $\mathbf{I} \in \mathscr{A}(\sigma)$, then $c_{\mathbf{J}} = 0$ $\forall \mathbf{J} \in \mathbf{I}_d \setminus \operatorname{star}(\sigma)$ and $c_{\mathbf{I}} = 1$.

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